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A Theory of Inclusions in Viscoelastic Materials*

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Recent years have witnessed a great strengthening of the interest taken by students of solid mechanics in the physical aspects of their problems. This has succeeded in recreating, to some extent, the exciting atmosphere of discovery associated in many people's mind with the age of the natural philosophers, the eighteenth and nineteenth centuries. After decades of usurpation by the pencil and paper of the mathematician, when physical assumptions were often acceptable provided they led to interesting analysis, experimentation has returned to its ordained position and the experimenter, rather than the theoretician, has become the most highly valued commodity on the labor market.

Two important spheres of activity of solid mechanics demonstrate well this trend: the theories of creep and of crack formation. In the former, nonlinear creep laws form the basis of the work of Hoff,[1] Rabotnov,[2] Arutiunian,[3] Kachanov,[4] and others in their endeavor to describe the long-range performance of such widely differing engineering materials as concrete, high polymers, and metals at elevated temperatures. In the latter, linear elastic theory combined with local plastic considerations and allowance for the forces of molecular cohesion provides the foundation on which Griffith,[5] Orowan,[6] Irwin,[7] Frenkel',[8] and Barenblatt[9] penetrate the mystery of failures in welded ships, turbine blades, and rock strata. A salient feature of both these activities is the role played by experiments and the consequent close cooperation between the investigators and nature.

This paper presents the results of a first attempt at understanding physical phenomena which in some respects bear a close relationship to those investigated in the above-mentioned theory of crack formation. In fact, like the ends of cracks, inclusions represent stress-raisers, and as the size of the inclusions decreases, one is bound to encounter conditions resembling those occurring around the ends of cracks. However, this is looking ahead, and at the present time consideration has been given only to isolated inclusions having dimensions which justify completely the use of small deformation theory. These inclusions are of the type the engineer introduces intentionally, for example, into concrete for the purpose of changing material strength. They occur also unintentionally as nonhomogeneities, for example, in products of the chemical industry as the result of insufficient control of reaction processes. In the case of elastic materials, problems involving inclusion phenomena have received attention for many years.[10-12]

As far as it has been possible to establish, the present treatment of isolated inclusions in linear viscoelastic materials is the first of its kind. It will show that the presence of an inclusion causes an "up-grading" of the order of the viscoelastic mechanism of the "bulk material," where it will be shown that the conditions at the interface between the bulk material and the inclusion play an important role. It will then be suggested that these results may be used to explain the great difficulties encountered in obtaining consistent experimental evidence for the strength properties of high polymers.

The use of linear, quasi-steady viscoelastic theory in work of this nature is well justified by the fact that the present study is concerned with short-term local effects in the vicinity of a smooth inclusion inside an otherwise homogeneously strained medium. In order to demonstrate the characteristic features of the problems under consideration and to avoid analytical complexities, most of the detailed work will refer to circular cylindrical inclusions in bodies in a state of plane strain, subject to uniaxial tension at infinity. Thus, the inclusion will play the part of a finite stress raiser.

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In view of the quasi-steady nature of the basic theory, the viscoelastic mechanism will be confined to the stress-displacement relations and it will be possible to utilize the results of the elastic theory. In the past, several methods have been developed for the purpose of obtaining viscoelastic solutions from corresponding elastic solutions.[3,13-16] A somewhat different approach will be used here which has been worked out recently and which in the case of plane strain utilizes the complex variable methods, established by Kolosov[16] and Muskhelishvili.[17]

Kirsch's[10] solution of the plane strain problem of uniaxial stress in an infinite solid with a circular cylindrical hole may be considered the first in a long chain of inclusion studies within the framework of classical elastic theory. There the most elegant results belong to Muskhelishvili[11] in the plane theory and to Goodier[12] in the three-dimensional theory. Their solutions form the basis for the present study.

Consider simultaneously the basic equations of the linear, quasi-static theories of elastic and viscoelastic plane strain for incompressible materials in their complex representation:[15]

$$\frac{\partial}{\partial \bar{z}}(\sigma_{22} - \sigma_{11} + 2i\sigma_{12}) = \frac{\partial}{\partial z}(\sigma_{11} + \sigma_{22}) \tag{1}$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = -4\mu \frac{\partial}{\partial z}(u_1 - iu_2)$$

$$P^p(\sigma_{22} - \sigma_{11} + 2i\sigma_{12}) = -2Q^q\left\{\frac{\partial}{\partial z}(u_1 - iu_2)\right\} \tag{2}$$

$$\frac{\partial}{\partial z}(u_1 + iu_2) + \frac{\partial}{\partial \bar{z}}(u_1 - iu_2) = 0 \tag{3}$$

where μ is the elastic shear modulus, $z = x_1 + ix_2$, and

$$P^p = \sum_{k=1}^{p} p_k \frac{\partial^k}{\partial t^k}\{\ \}$$

$$Q^q\{\ \} = \sum_{k=1}^{q} q_k \frac{\partial^k}{\partial t^k}\{\ \} \tag{4}$$

are viscoelastic operators. If one of the formal methods of viscoelasticity[13,14] is used, the quantity 2μ in the elastic solution is replaced by Q/P, if necessary after application of the Laplace transformation. The viscoelastic solution is then obtained by inversion of the Laplace transform or by solution of ordinary differential equations. However, in this context, it is to be noted that the elastic theory usually employs an Airy stress function which is the solution of a biharmonic equation, arising, in general, from the following equation in the case of absence of body forces (X_1, X_2):

$$(\lambda + 2\mu)\nabla^4 U = -2(\lambda + \mu)\left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2}\right) \tag{5}$$

In the viscoelastic case, the operator corresponding to the factor $(\lambda + 2\mu)$ may not be removed by division.

With consideration of the operator character of P^p, Q^q, the general solutions of eqs. (1)–(3) may be written in the form:[15]

$$\sigma_{11} + \sigma_{22} = 2[\varphi'(z) + \overline{\varphi'(z)}] \tag{6}$$

$$P^p\{\sigma_{11} + \sigma_{22}\} = 2[\varphi'(z) + \overline{\varphi'(z)}]$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2[\bar{z}\varphi''(z) + \psi'(z)] \tag{7}$$

$$P^p\{\sigma_{22} - \sigma_{11} + 2i\sigma_{12}\} = 2[\bar{z}\varphi''(z) + \psi'(z)]$$

$$2\mu(u_1 + iu_2) = \varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \tag{8}$$

$$Q^q\{u_1 + iu_2\} = \varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}$$

where, as in the elastic theory, $\varphi(z)$, $\psi(z)$ are functions of z which are holomorphic in the region occupied by the body.[11]

For elastic materials, the functions φ, ψ, describ-

TABLE I

$$\varphi(z) = \frac{\Pi}{4}\left(z + \frac{\beta R^2}{z}\right)$$

$$\psi(z) = -\frac{\Pi}{2}\left(z + \frac{\gamma R^2}{z} + \frac{\delta R^4}{z^3}\right)$$

Materials[a] Inclusion	Bulk	Attachment	β	γ	δ	Limits of solution for $r = R$, σ_{rr} = radial stress
Void	Elastic	Zero	2	1	−1	—
Rigid	Elastic	Attached	−2	0	1	—
Rigid	Elastic	Unattached	−1	0	0	$\sigma_{rr} < 0$
Elastic	Elastic	Attached	$\frac{2(\mu_1 - \mu_2)}{\mu_1 + \mu_2}$	0	$\frac{\mu_2 - \mu_1}{\mu_1 + \mu_2}$	—
Elastic	Elastic	Unattached	$\frac{2\mu_1 - \mu_2}{\mu_1 + \mu_2}$	0	$\frac{-\mu_1}{\mu_1 + \mu_2}$	$\sigma_{rr} < 0$

[a] μ_1 = bulk material; μ_2 = inclusion material.

TABLE II

$\sigma_{11} + \sigma_{22} = E + FR^2/r^2 \cos 2\theta,$

$\sigma_{22} - \sigma_{11} = [G(R^2/r^2) + H(R^4/r^4) + K] \cos 2\theta + L(R^2/r^2)$

Inclusion	Bulk	Mode of attachment	E	F	G	H	K	L
Void	Elastic	Zero	Π	-2Π	2Π	-3Π	$-\Pi$	Π
Rigid	"	Attached	Π	2Π	-2Π	3Π	$-\Pi$	0
Rigid	"	Unattached	Π	Π	$-\Pi$	0	$-\Pi$	0
Elastic	"	Attached	Π	$\dfrac{2\Pi(\mu_2 - \mu_1)}{\mu_1 + \mu_2}$	$\dfrac{-2\Pi(\mu_2 - \mu_1)}{\mu_1 + \mu_2}$	$\dfrac{3\Pi(\mu_2 - \mu_1)}{\mu_1 + \mu_2}$	$-\Pi$	0
Elastic	"	Unattached	Π	$\dfrac{-\Pi(2\mu_1 - \mu_2)}{\mu_1 + \mu_2}$	$\dfrac{\Pi(2\mu - \mu_2)}{\mu_1 - \mu_2}$	$\dfrac{-3\Pi\mu_1}{\mu_1 + \mu_2}$	$-\Pi$	0
Void	Maxwell	Zero	$\dfrac{1}{p_1} I_1(\tau_1)$	$\dfrac{-2}{p_1} I_1(\tau_1)$	$\dfrac{2}{p_1} I_1(\tau_1)$	$\dfrac{-3}{p_1} I_1(\tau_1)$	$\dfrac{-1}{p_1} I_1(\tau_1)$	$\dfrac{1}{p_1} I_1(\tau_1)$
Rigid	"	Attached	$\dfrac{1}{p_1} I_1(\tau_1)$	$\dfrac{2}{p_1} I_1(\tau_1)$	$\dfrac{-2}{p_1} I_1(\tau_1)$	$\dfrac{3}{p_1} I_1(\tau_1)$	$\dfrac{-1}{p_1} I_1(\tau_1)$	0
Rigid	"	Unattached	$\dfrac{1}{p_1} I_1(\tau_1)$	$\dfrac{1}{p_1} I_1(\tau_1)$	$\dfrac{-1}{p_1} I_1(\tau_1)$	0	$\dfrac{-1}{p_1} I_1(\tau_1)$	0
Void	Voigt	Zero	Π	-2Π	2Π	-3Π	$-\Pi$	Π
Rigid	"	Attached	Π	2Π	-2Π	3Π	$-\Pi$	0
Rigid	"	Unattached	Π	Π	$-\Pi$	0	$-\Pi$	0
Elastic	Maxwell	Attached	$\dfrac{1}{p_1} I_1(\tau_1)$	$\dfrac{1}{p_1\tau_2}\{2(2\tau_1 - \tau_2) \times I_1(\tau_1) + \dfrac{4}{\tau_2}(\tau_2 - \tau_1)I_2(\tau_1,\tau_2)\}$	$\dfrac{1}{p_1\tau_2}\{2(\tau_2 - \tau_1)I_1(\tau_1) - \dfrac{4}{\tau_2}(\tau_2 - \tau_1) \times I_2(\tau_1,\tau_2)\}$	$\dfrac{1}{p_1\tau_2}\{3(\tau_1 - \tau_2)I_1(\tau_1) + \dfrac{6}{\tau_2}(\tau_2 - \tau_1) \times I_2(\tau_1,\tau_2)\}$	$\dfrac{-1}{p_1}\{I_1(\tau_1) - \dfrac{2}{\tau_2} I_2(\tau_1,\tau_2)\}$	0
Elastic	Maxwell	Unattached	$\dfrac{1}{p_1} I_1(\tau_1)$	$\dfrac{1}{p_1\tau_2}\{(3\tau_1 - 2\tau_2) \times I_1(\tau_1) + \dfrac{3}{\tau_2}(\tau_2 - \tau_1) \times I_2(\tau_1,\tau_2)\}$	$\dfrac{1}{p_1\tau_2}\{(2\tau_2 - 3\tau_1)I_1(\tau_1) - \dfrac{1}{\tau_2}(3\tau_2 - 3\tau_1) \times I_2(\tau_1,\tau_2)\}$	$\dfrac{1}{p_1\tau_2}\{(3\tau_1 - 3\tau_2) \times I_1(\tau_1) + \dfrac{1}{\tau_2}(3\tau_2 - 3\tau_1)I_2(\tau_1,\tau_2)\}$	$\dfrac{-1}{p_1}\{I_1(\tau_1) + \dfrac{1}{\tau_2} I_2(\tau_1,\tau_2)\}$	0
Elastic	Voigt	Attached	Π	$-2\Pi + 4\left(\dfrac{1}{\tau_2} - \dfrac{1}{\tau_1}\right) \times I_1(\tau_2)$	$2\Pi + 4\left(\dfrac{1}{\tau_1} - \dfrac{1}{\tau_2}\right) \times I_1(\tau_2)$	$-3\Pi - 6\left(\dfrac{1}{\tau_1} - \dfrac{1}{\tau_2}\right) \times I_1(\tau_2)$	$-\Pi$	0
Elastic	Voigt	Unattached	Π	$-2\Pi + 3\left(\dfrac{1}{\tau_2} - \dfrac{1}{\tau_1}\right) \times I_1(\tau_2)$	$2\Pi + 3\left(\dfrac{1}{\tau_1} - \dfrac{1}{\tau_2}\right) \times I_1(\tau_2)$	$-3\Pi - 3\left(\dfrac{1}{\tau_1} - \dfrac{1}{\tau_2}\right) \times I_1(\tau_2)$	$-\Pi$	0

ing the state of stress and deformation in the bulk material surrounding circular inclusions, may be represented in the form[11]

$$\varphi(z) = \frac{\Pi}{4}\left(z + \frac{\beta R^2}{z}\right)$$

$$\psi(z) = -\frac{\Pi}{2}\left(z + \frac{\gamma R^2}{z} + \frac{\delta R^4}{z^3}\right) \tag{9}$$

where Table I summarizes the values of the parameters β, γ, δ for various types of inclusions.

The differential equations in time for the viscoelastic solutions are now readily deduced from these elastic solutions, since the process of satisfying the stress and displacement conditions at the interfaces involves only simple algebraic operations which are permissible for the operators P^p, Q^q. It is readily seen that in certain cases this procedure leads to ordinary differential equations with constant coefficients the order of which is determined by the sum of the orders of the operators P^p and Q^q; in other words, the order of the model of the bulk material without inclusion is specified by that of the operators P^p, Q^q, while the order of the material with inclusion is specified by that of the product of these operators.

For example, consider the two simplest models for viscoelastic materials:

the Maxwell model with

$$P' = p_0 + p_1\,\partial/\partial t \qquad (10)$$
$$Q' = \partial/\partial t$$

the Voigt model with

$$P_0 = 1 \qquad (11)$$
$$Q' = q_0 + q_1\,\partial/\partial t$$

In these two cases, the characteristic times in the presence of an elastic inclusion are for the Maxwell model:

$$\tau_1 = p_1/p_0 \qquad (12)$$
$$\tau_2 = [(^1/_2p_1 + u_2)/\mu_2]\,\tau_1$$

for the Voigt model:

$$\tau_1 = q_1/q_0 \qquad (13)$$
$$\tau_2 = [q_0/2/(q_0/2 + \mu_2)]\tau_1$$

where in each case τ_1 corresponds to the relaxation and retardation time, respectively, of the homogeneous material. The quantities $^1/_2p_1$ and $q_0/2$, respectively, play the role of the shear moduli of the materials, and hence will be assumed to be of the same order of magnitude as the shear modulus μ_2 of the elastic inclusion. It follows that the Maxwell material with an inclusion acquires a second, substantially larger relaxation time, while the Voigt material acquires a second, substantially smaller retardation time.

For example, assuming for both materials that

$$\tau_1 = 0.01$$

and

$$\mu_2 = {}^1/_2\mu_1,\ \mu_1,\ 2\mu_1$$

one finds for the two materials the corresponding pseudo-times:

Maxwell material:

$$3\tau_1,\ 2\tau_1,\ {}^3/_2\tau_1 \qquad \text{(A)}$$

Voigt materials:

$$^2/_3\tau_1,\ {}^1/_2\tau_1,\ {}^1/_3\tau_1$$

Introducing the two integrals

$$I_1(\tau_1) = \int_0^t e^{(s-t)/\tau_1}\,\Pi(s)ds \qquad (14)$$
$$I_2(\tau_1,\tau_2) = \int_0^t e^{(\sigma-t)/\tau_1}\,I_1(\tau_2)d\sigma$$

the general elastic and viscoelastic solutions, i.e., the expressions for the stresses and displacements in the bulk material for the cases of Table I, can be presented in the form of Table II with Π representing the loading varying with time.

Now let

$$\Pi(t) = \Pi_0(1 - e^{-t/\gamma}) \qquad (15)$$

with γ a parameter determining the initial rate of loading

$$(d\Pi/dt)_{t=0} = \Pi_0/\gamma \qquad (16)$$

Figure 1 shows that loading function for $\gamma = 0.05$, 0.10, 1.0, and 10.0 and demonstrates its usefulness for the present investigation.

The case of the elastic, attached inclusion will now be considered in detail for the Maxwell material with the relaxation time $\tau_1 = 0.01$ and an inclusion with $\mu_2 = {}^1/_2\mu_1$, so that $\tau_2 = 0.03$; and for the Voigt material with the retardation time $\tau_1 = 0.03$ and an inclusion with $\mu_2 = 2\mu_1$, so that $\tau_2 = 0.01$.

Figures 2–5 demonstrate the changes in the stress distributions around the inclusion in the two materials as functions of the time as well as of the distance from the inclusion. These curves illustrate

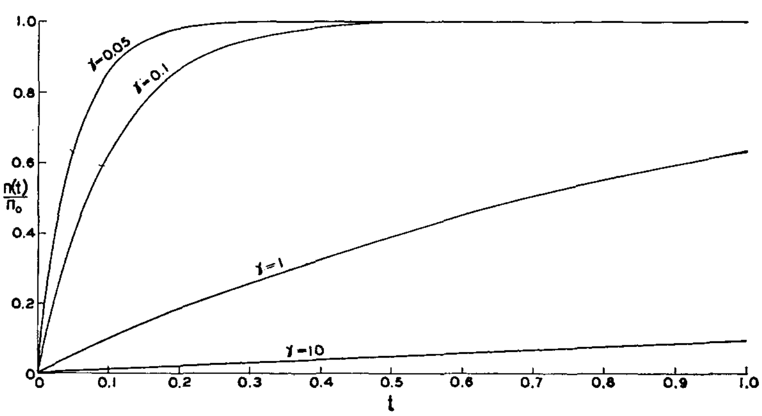

Fig. 1. Loading histories.

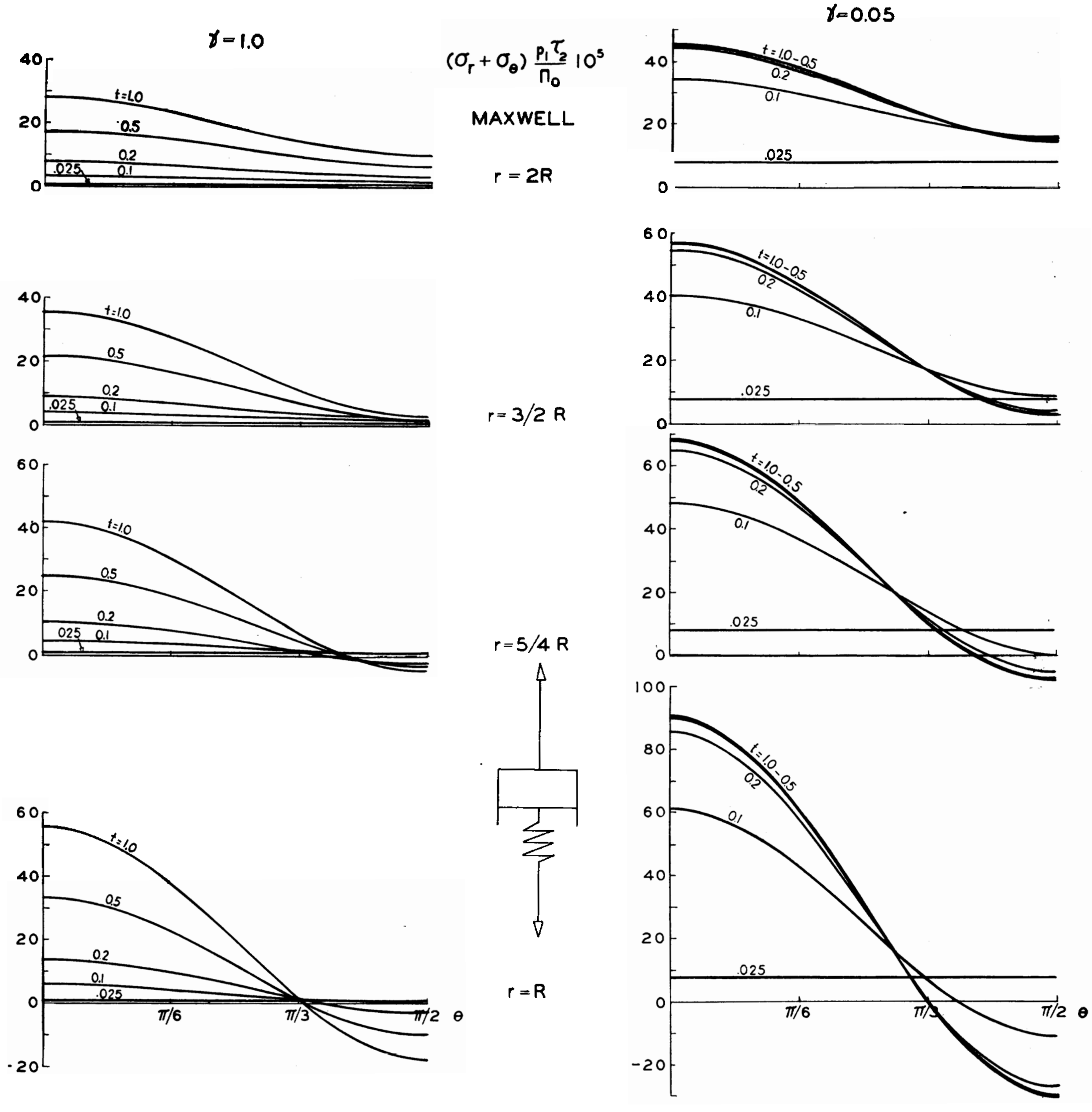

Figure 2.

well the typical behavior of the respective materials under two-dimensional conditions.

Figure 6 shows the time histories of the sum and difference of the direct stresses at a point on the interface between the same bulk materials and attached and unattached inclusions. It is important to note here that the curves for unattached inclusions have validity only while $\sigma_r < 0$, since otherwise detachment occurs and the imposed boundary conditions are no longer applicable. The presence of the second characteristic time is demonstrated by the greater complexities of these time histories which exhibit extreme values and points of inflection. Indeed, in some cases it can lead to a reversal of the stresses during the early stages of the loading process.

Clearly, in cases of several inclusions lying closely together, the above effects will accumulate and lead to a finite number of characteristic times. As the number of inclusions increases, one will approach the stage when one has to deal with a spectrum of such times. This is in agreement with the

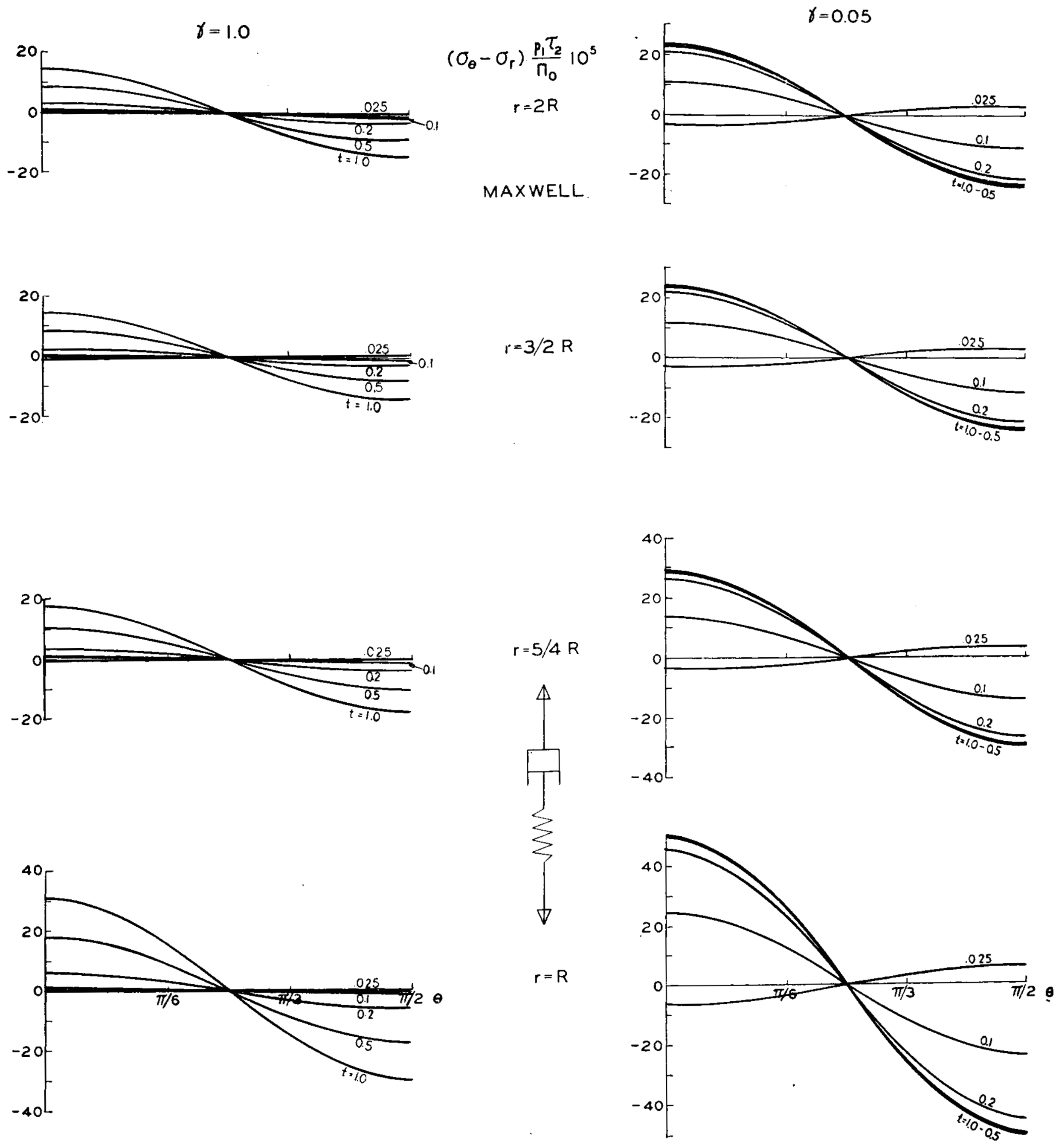

Figure 3.

conclusion reached by Kolsky and Shi[18] that physical materials cannot be represented over their entire range of loading by low order operators. It might be argued that their work is concerned with homogeneous materials, and that therefore their conclusions cannot be explained by the presence of inclusions. At this stage, it can only be suggested that the molecular structure of many viscoelastic materials of interest today, i.e., the macromolecules of high polymers, tend to form continua which might be conceived to be homogeneous except for interspersed stress raisers of the type considered here.

At this stage it becomes natural to think in terms of a close relationship between the present work and Barenblatt's work on crack formation.[9] As the

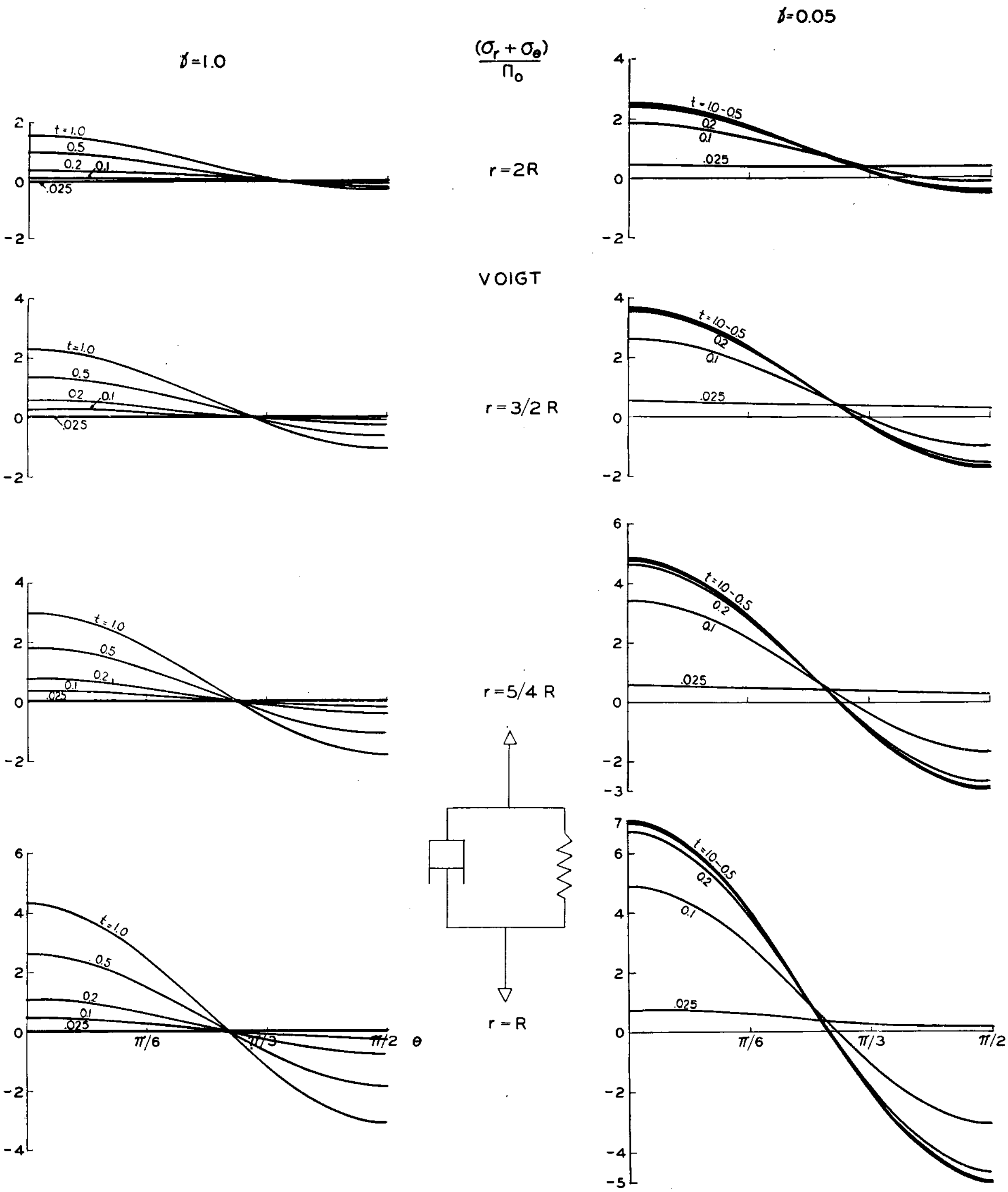

Figure 4.

size of the stress raisers decreases, it will become necessary to take into account the forces of molecular cohesion, and this establishes the need for experimental work on such forces in high polymers. As in the work on crack formation, it is clear that the size of the inclusions, i.e., in the above cases, the diameter *R*, is an indefinite parameter, a circumstance which can hardly be expected to prevail over the entire range of inclusion sizes.

Extensions of this work to three dimensional in-

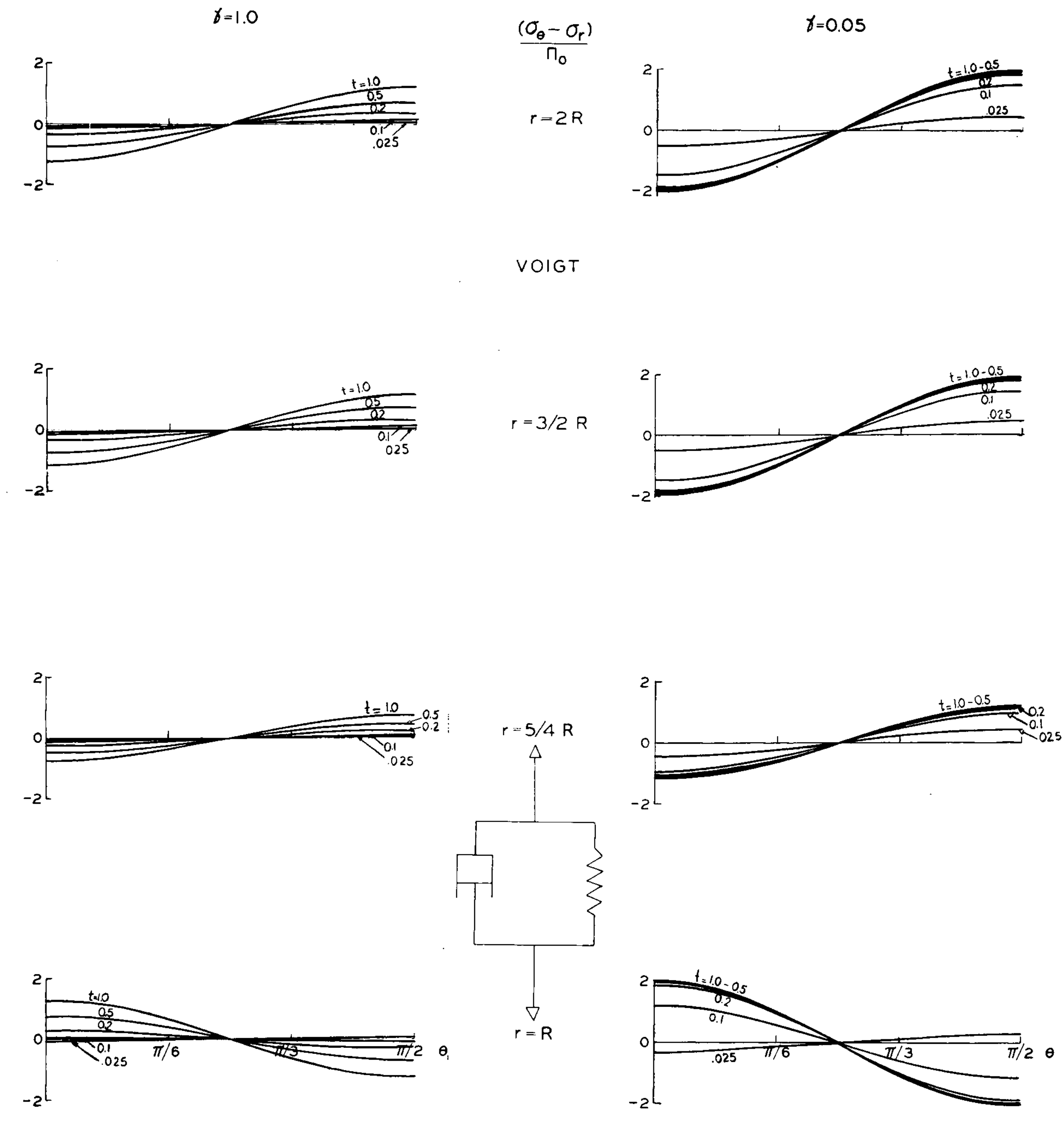

Figure 5.

clusions and to compressible materials greatly complicate the analysis without increasing fundamental knowledge. The same observation applies to viscoelastic inclusions. In the case of plane strain for compressible materials eqs. (3), and (6)–(8) are replaced[15] by

$$\sigma_{11} + \sigma_{22} = \frac{2\mu_1}{1 - 2\sigma_1}\left[\frac{\partial}{\partial z}(u + iv) + \frac{\partial}{\partial \bar{z}}(u - iv)\right]$$

$$R^r(\sigma_{11} + \sigma_{22}) = S^s\left\{\frac{\partial}{\partial z}(u + iv) + \frac{\partial}{\partial \bar{z}}(u - iv)\right\} \tag{3'}$$

$$\sigma_{11} + \sigma_{22} = 2[\varphi'(z) + \overline{\varphi'(z)}] \tag{6'}$$

$$(Q^qR^r + {}^{\boldsymbol{\cdot}}P^pS^s)\{\sigma_{11} + \sigma_{22}\} = 2[\varphi'(z) + \overline{\varphi'(z)}]$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2[\bar{z}\varphi''(z) + \psi'(z)] \tag{7'}$$

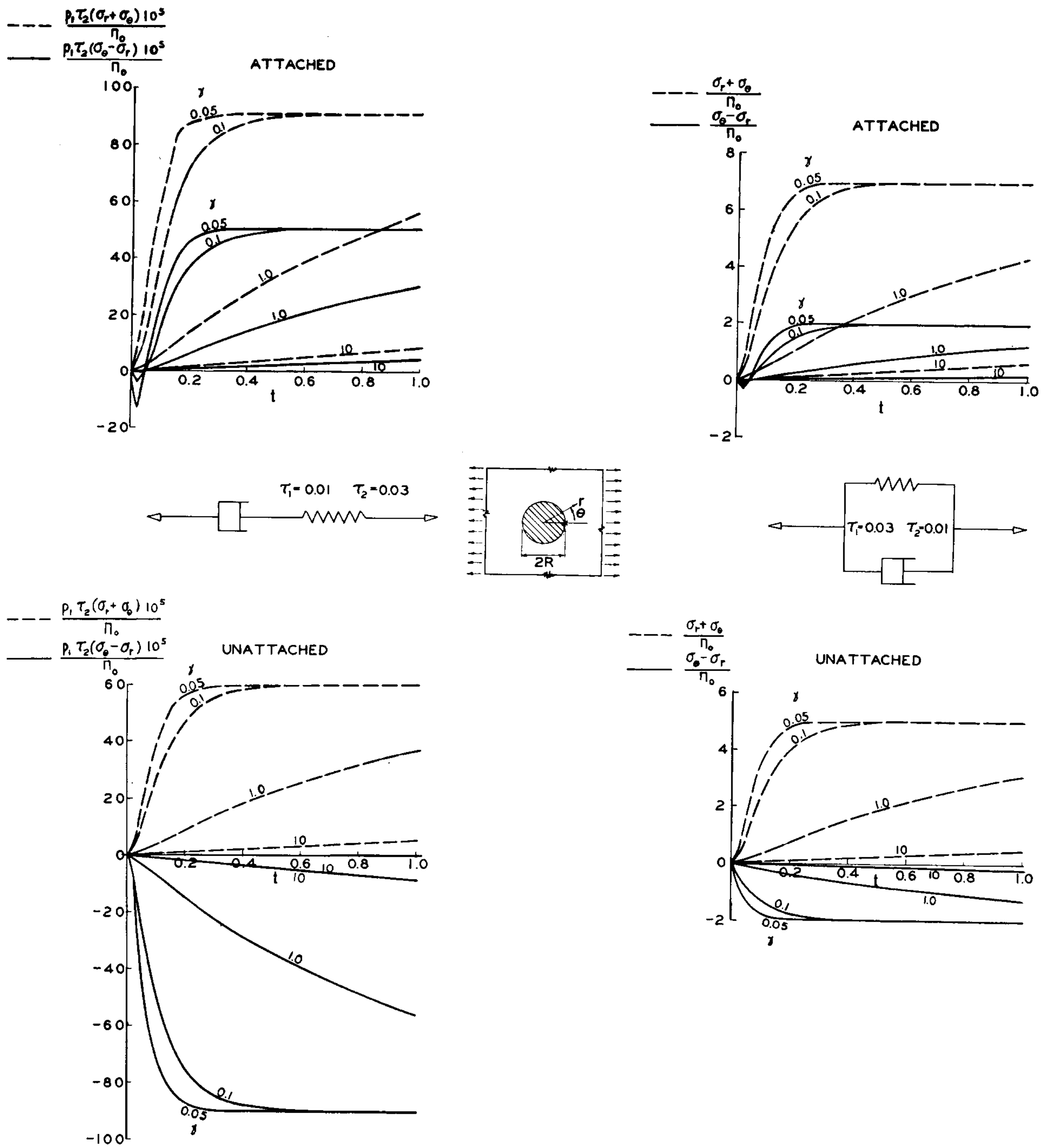

Figure 6.

$$(Q^qR^r + P^pS^s)\{\sigma_{22} - \sigma_{11} + 2i\,\sigma_{12}\} = 2[\bar{z}\varphi''(z) + \psi'(z)]$$

$$2\mu_1(u_1 + iu_2) = (3-4\sigma_1)\varphi(z) - \overline{z\varphi'(z)} - \overline{\psi(z)} \quad (8')$$

$$Q^qS^s(Q^qR^r + P^pS^s)\{u + iv\} = (2Q^qR^r + P^qS^s) \times \{\varphi(z)\} - P^qS^s\{\overline{z\varphi'(z)} + \overline{\psi(z)}\}$$

As a result, the order of the ordinary differential equations in time which lead from the elastic to the viscoelastic solutions is increased and hence the viscoelastic stress histories exhibit the presence of additional pseudo-characteristic times. In fact, for the cases of attached and unattached inclusions in Maxwell and Voigt materials, this number of acquired relaxation and retardation times rises to four, and it is no longer possible to give simple formulae for their values in terms of the material constants.

The study of three dimensional spherical inclusions on the basis of the elastic results of Goodier[12] leads to an even greater complexity which has precluded completion, and hence publication in this report, of the pertinent analysis. Since the work of Goodier[12] is based on the Navier equations and their solutions in terms of spherical harmonics, the formal procedure of establishing formulae for viscoelastic problems differs from that adopted in the case of plane strain, where effectively a stress function had been employed. In the two dimensional case, the stresses are determined first and the displacements follow from the strain-stress relations. In the three-dimensional case the displacements are evaluated from the equations

$$(\lambda + \mu)(\partial\theta/\partial x_i) + \mu\nabla^2 u_i = 0 \qquad (17)$$

or

$$[(2P^pS^s + Q^qR^r)](\partial\theta/\partial x_i) + 3Q^qR^r\ \nabla^2 u_i = 0$$

where

$$\theta = \partial u_k/\partial x_k \qquad (17)$$

and the stresses must be obtained from the stress–strain relations

$$\sigma_{ij} = \lambda\theta\delta_{ij} + \mu[(\partial u_i/\partial x_j) + (\partial u_j/\partial x_i)] \qquad (18)$$

$$P^pR^r\sigma_{ij} = {}^1/_3(P^pS^s - Q^qR^r)\theta\delta_{ij} + (Q^qR^r/2)[(\partial u_i/\partial x_j) + (\partial u_j/\partial x_i)]$$

Detailed results on these and related problems will be discussed in a later report.

In conclusion, the deductions of the present investigation may be summarized as follows. As in elastic materials, an inclusion in a viscoelastic material plays the role of a stress raiser which, in the case of smooth boundaries, is of finite order. In addition, they also change locally the time dependent behavior of the bulk material which appears to acquire additional characteristic times, normally associated with higher order models. As a consequence, it may be suggested that viscoelastic materials with impurities, distributed throughout at random, are bound to require either high order models or continuous spectra of characteristic times for an adequate description of their response in time and space.

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Synopsis

Two important groups of viscoelastic materials may, in practical applications, be characterized by the presence of inclusions. In high polymers, the control of the production of homogeneous batches presents often unsurmountable difficulties and hence unintended regions of nonhomogeneity will result. In concrete structures, reinforcements are often introduced to ensure strength in tension, and these inclusions, while intentional, are bound to change the behavior of the bulk material. This investigation of the effects of circular cylindrical and spherical inclusions in infinite two- and three-dimensional bodies, subject to uniaxial loading at infinity, has been based on linear viscoelastic theory. It has been assumed that the inclusions are either rigid, perfectly elastic or viscoelastic and that their boundaries are free or welded to the surrounding material. In all these cases, the behavior of the viscoelastic material surrounding the inclusions will differ essentially from that of the material without inclusions. For example, it may be shown that a single parameter material will exhibit several retardation or relaxation times. In view of this situation, the interpretation of experimental test results on viscoelastic materials as well as the practical usefulness of linear viscoelastic theory appear in a new light which is discussed on the basis of the available information.

Résumé

Deux groupes importants de matériaux viscoélastiques peuvent, pour les applications pratiques, être caractérisés par la présence d'inclusions. Dans le cas des hauts polymères le contrôle de la production lors de la préparation en bains homogènes présente souvent des difficultés insurmontables avec comme conséquence involontaire la présence des régions non-homogènes. Dans les structures concrêtes des renforcements sont souvent introduits pour augmenter la force de tension et ces inclusions, qui sont voulues, sont ajoutées pour modifier le comportement de la masse de matériaux. Les recherches, sur les effets d'inclusions circulaires, cylindriques et sphériques dans des objets bi- et tridimensionnels, soumis à une charge uniaxiale à l'infini, ont été basées sur la théorie viscoélastique linéaire. On a supposé que les inclusions sont soit rigides, parfaitement élastiques ou viscoélastiques et que leurs frontières sont libres on soudées au matériel qui les entoure. Dans tous ces cas, le comportement de la substance viscoélastique entourant les inclusions diffèrera essentiellement du comportement de la substance sans inclusions. On peut montrer par exemple qu'un matériau caractérisé par un paramètre unique manifestera plusieurs temps de retard et de relaxation. D'après ces données, l'interprétation des résultats des essais expérimentaux sur des substances viscoélastiques aussi bien que l'utilité pratique d'une théorie viscoélastique linéaire apparaissent sous un angle nouveau qui est discuté sur la base des renseignments disponibles.

Zusammenfassung

Zwei wichtige Gruppen von viskoelastischen Materialien können bei praktischer Anwendung durch das Vorhandensein von Einschlüssen charakterisiert werden. Bei Hochpolymeren bietet die Kontrolle der Herstellung homogener Massen oft unüberwindbare Hindernisse und es können daher unbeabsichtigt nichthomogene Bereiche entstehen. Bei Betonkonstruktionen werden oft zur Festigkeitserhöhung Verstärkungen eingeführt und diese absichtlichen Einschlüsse werden notwendigerweise das Verhalten des ursprünglichen Materials verändern. Der vorliegenden Untersuchung des Einflusses von zylindrischen und kugelförmigen Einschlüssen in zwei- oder dreidimensionalunendlichen Körpern, die einer uniaxialen Belastung unterwortfen waren, liegt die lineare Viskoelastizitätstheorie zugrunde. Es wurde angenommen, dass die Einschlüsse entweder starr, vollkommen elastisch oder viskoelastisch sind und dass ihre Oberfläche frei oder mit dem umgebenden Material verschmolzen ist. In allen diesen Fällen wird sich das Verhalten des viskoelastischen Materials in der Umgebung der Einschlüsse von dem des Materials ohne Einschlüsse wesentlich unterscheiden. Zum Beispiel kann gezeigt werden, dass ein Ein-Parameter-Material einige Verzögerungs- und Relaxationszeiten aufweisen wird. In Hinblick auf diese Situation erscheint sowohl die Interpretation der Testergebnisse an viskoelastischen Materialien als auch die praktische Verwendbarkeit der linearen Viskoelastizitätstheorie in einem neuen Licht; eine Diskussion auf Grund der vorhandenen Angaben wird durchgeführt.

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